

Dynamics to equilibrium in Network Games: individual behavior and global response

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Abstract

Various social contexts ranging from public goods provision to information collection can be depicted as games of strategic interactions, where a player's well-being depends on her own action as well as on the actions taken by her neighbors. Whereas much attention has been devoted to the identification and characterization of Bayes-Nash equilibria of such games, in this work we look at strategic interactions from an evolutionary perspective. Starting from a recent mean-field analysis of the evolutionary dynamics in these games, here we present results of numerical simulations designed to find out whether Nash equilibria are accessible by adaptation of players' strategies, and in general to find the attractors of the evolution. Simulations allow us to go beyond a global characterization of the cooperativeness of the equilibria and probe into the individual behavior. We find that when players imitate each other, the evolution does not reach Nash equilibria and, worse, leads to very unfavorable states in terms of welfare. On the contrary, when players update their behavior rationally, they self-organize into a rich variety of Nash equilibria, where individual behavior and payoffs are shaped by the nature of the game, the structure of the social network and the players' position within the topology. Our results allow us to assess the validity of the mean-field approaches and also show qualitative agreement with theoretical predictions for equilibria in the context of one-shot games under incomplete information.

1 Introduction

In a range of social and economic interactions (public goods provision, job search, political alliances, trade, friendships, information collection, and so on) an individual's welfare depends both on her own actions and on the actions taken by her interacting partners. Much effort has been devoted to understand how the pattern of social connections shapes the choices that individuals make and the payoffs that they can earn [1–7] (see [8–10] for an overview of the field). On the theoretical side, the traditional approach of identifying the

Bayes-Nash equilibria in one-shot network games is problematic due to the existence of multiple equilibria (even for very small network sizes), which imply a huge range of possible outcomes. Nevertheless, it has been shown that the problem of multiplicity can be resolved sometimes by the introduction of incomplete information [11, 12]—which in the context of network games means having only a local knowledge of the network: a player is aware of the number of her connections (her degree) but not of the degrees of the others. The key point is that when players have limited information about the network they are unable to condition their behavior on its fine details, and this leads to a significant simplification and sharpening of equilibrium predictions. A general framework for the study of multiplayer games on networks under incomplete information has been recently developed [13, 14]; in particular, Galeotti et al. [14] proved the existence of Bayes-Nash equilibria involving strategies that are monotonic with respect to players’ degrees and symmetric, *i.e.*, with all players of the same degree choosing the same strategy.

Here, instead of focusing on equilibrium strategies played once and for all, we consider iterated network games within an evolutionary framework [15]: players’ own actions are described by a strategy which is subject to an evolutionary process [16, 17]. It is important to note that this is different from the stochastic stability approach presented for this type of games in [18], where the focus is on the different time regimes of the evolution. In fact, our framework represents a way to select among the large multiplicity of equilibria that exist in network games in a spirit very close to that of biological evolution. In particular, we consider two main mechanisms for players to adapt their strategy: imitation and rational deduction. Moreover, we consider two representative social network structures (Erdős-Rényi random graphs [19] and scale-free networks [20]) and, as in [14], two canonical types of interaction: the best-shot game and the coordination game, as representatives for strategic substitutes and strategic complements, respectively [21]. We run a numerical simulation program in order to understand which Nash equilibria are dynamically accessible (by adaptation of players’ strategies), and in general to identify the strategy configurations that are the attractors of such dynamics—either Nash equilibria or, possibly, other types of stationary states. As a reference, we consider the analytical results we obtained in [22], where we studied this issue by means of two mean field approaches, the usual, homogeneous one, and its heterogeneous version [23]. As shown below, the numerical simulations allow us to assess the accuracy of the pictures provided by the two approaches and, furthermore, to gain more insight on the individual behavior of the different types of players. Additionally, we compare our approach with the theoretical framework of Galeotti et al. [14] for one-shot games under incomplete information. We show that the Nash equilibria predicted by their theory generally do not coincide with the ones obtained by evolutionary dynamics; nevertheless, these equilibria possess qualitatively the same features of those resulting from evolution, in terms of players’ strategies with respect to their neighborhood structure.

Our work is organized as follows. We begin by introducing in Section 2 the games we will be considering, followed by a presentation of the two evolutionary dynamics in Section 3. Section 4, the main part of the paper, presents the results of numerical simulations, which, for reference, are accompanied by a brief review of the mean field results in [22]. Finally, Section 5 concludes the paper with a summary of the results and a discussion of how they compare to the different theoretical frameworks.

2 Games and Equilibria

This section presents two simple games played on networks, reflecting strategic substitutes and strategic complements, respectively [21]. These two cases represent alternative scenarios on how a player’s payoff is affected by the actions of others, covering many of the game-theoretic applications studied in the economic literature. In particular, strategic substitutes encompass many scenarios that allow for free riding or have a

public good structure of play, whereas strategic complements arise whenever the benefit that an individual obtains from undertaking a given behavior is greater as more of her partners do the same.

Below, we first recall the framework by Galeotti et al. [14] under the assumption of incomplete information, which is natural in many circumstances. Consider a society of n agents, placed on the nodes of a social network. The links between agents reflect social interactions, and connected agents are said to be *neighbors*. Every individual must choose independently an action in $X = \{0, 1\}$, where action 1 may be interpreted as *cooperating* and action 0 as not doing so—or *defecting*. To define the payoffs, let x_i be the action chosen by agent i , N_i the set of i 's neighbors, $x_{N_i} = \sum_{j \in N_i} x_j$ the aggregate action in N_i , and $y_i = x_i + x_{N_i}$. There is a cost c , where $0 < c < 1$, for choosing action 1, while action 0 bears no cost.

Strategic Substitutes: Best-shot game

The payoff function takes the form

$$\pi_i = \Theta_H(y_i - 1) - c x_i, \quad (1)$$

where $\Theta_H(\cdot)$ is the Heaviside step function $\Theta_H(x) = 1$ if $x \geq 0$ and $\Theta_H(x) = 0$ otherwise. Strategic substitutes thus represent an anti-coordination game: a player would prefer that someone of her neighbors takes action 1 (rather than taking the action herself), but she would be willing to take action 1 if nobody in the neighborhood does. In general, a context in which players have complete information on the social network allows for a very rich set of Nash equilibria of the game [13, 14], where the relation between network connections, equilibrium actions and payoffs may exhibit very different patterns (even when all agents of the same degree choose the same actions). Things change, however, by relaxing the assumption of complete information—as shown in [14]. The assumption is that each player knows her own degree k' and the probability distribution $P(k|k')$ of the degree k of her neighbors, which for uncorrelated networks (degrees of neighboring nodes are independent) reads

$$P(k|k') = kP(k)/\bar{k}, \quad (2)$$

where $P(k)$ is the degree distribution of the network and $\bar{k} = \sum_k kP(k)$ the average connectivity. Under these conditions, Galeotti et al. [14] study the game within the framework of Bayesian games and show that a player's (pure) strategy $\sigma \in X$ depends only on the degree k of the player. If an agent of degree k chooses action 1 in equilibrium, it must be because she does not expect that any of her neighbors will choose action 1. Therefore, in an uncorrelated network, an agent of degree $k - 1$ faces a lower likelihood of an arbitrary neighbor choosing the action 1, and would be best responding with action 1 as well. In particular [14], any Nash equilibrium is characterized by a threshold τ , the smallest integer for which

$$1 - \left[1 - \sum_{k=1}^{\tau} \frac{kP(k)}{\bar{k}} \right]^{\tau} \geq 1 - c, \quad (3)$$

and an equilibrium σ must satisfy $\sigma(k) = 1$ for all $k < \tau$, $\sigma(k) = 0$ for all $k > \tau$ and $\sigma(\tau) \in \{0, 1\}$ (i.e. $\sigma(k)$ is non-increasing).

Strategic Complements: Coordination game

The payoff function here takes the form

$$\pi_i = (\alpha x_{N_i} - c) x_i, \quad (4)$$

where $0 < \alpha < c$. Strategic complements thus represent a coordination game. As for substitutes, in the case of complete information there are generally many equilibria (including the case $x_i = 0 \forall i$, which represents full defection). Also here, by making the assumption that each player is only informed of her own degree and has independent beliefs on the degrees of neighbors, it is possible to find much more definite predictions with regard to equilibrium behavior [14]. In particular, independence of neighbor degrees implies that the probability that a random neighbor chooses the action 1 cannot depend on one's own degree (a player's neighbors do not know her degree and, consequently, they cannot know whether or not it will be convenient for her to chose action 1), and the expectation of the sum of actions x_{N_i} of any agent i with $|N_i| = k$ neighbors is increasing in k . The structure of payoffs then assures that if a degree k agent is choosing the action 1 in equilibrium, any agent of degree greater than k must be best responding with the action 1 as well, as she should have as many cooperating neighbors as an agent of degree $k - 1$. Therefore every equilibrium is characterized by an integer threshold τ [14] such that

$$\alpha(\tau - 1) \sum_{k=\tau}^{n-1} \frac{kP(k)}{\bar{k}} < c \quad \text{and} \quad \alpha\tau \sum_{k=\tau}^{n-1} \frac{kP(k)}{\bar{k}} \geq c. \quad (5)$$

The equilibrium satisfies $\sigma(k) = 0$ for all $k < \tau$, $\sigma(k) = 1$ for all $k > \tau$ and $\sigma(\tau) \in \{0, 1\}$ (in particular, $\sigma(k)$ is non-decreasing).

Summing up, for equilibrium strategies played once and for all under complete network information there is no systematic relation between social networks and individual behavior and payoffs. By contrast, under incomplete network information, both in games of strategic substitutes and of strategic complements, there is a clear cut relation between networks and individual behavior, as decisions can in general be inferred from degrees.

3 Evolutionary Dynamics

Whereas the previous section dealt with Bayes-Nash equilibria of one-shot strategic games under the settings of complete and incomplete information, here we move to our contribution, namely an evolutionary scenario in which agents play multiple instances of the game, and can adapt their strategy hoping to increase their payoffs. We resort to an evolutionary game-theoretical approach in which agents do not make strategic considerations about the network's global structure and their position within it; instead, they base their decision on their own actions/payoffs and on those of neighbors, as observed in the past. Our goal is to study the resulting dynamics of the system as well as its equilibrium configurations, and check whether the system can reach a Nash equilibrium, or what is the nature of the steady state if not Nash.

The evolutionary system can be described as follows. Starting with a fraction ρ_0 of players randomly chosen to undertake action $x = 1$, at each round t of the game: players collect their payoff $\pi^{(t)}$ —given by Eq.(1) and Eq.(4) respectively for strategic substitutes and complements; then a fraction q of players update their strategy. We consider two different mechanisms for strategy updating.

Proportional Imitation (PI) [24] — It represents a rule of imitative nature in which player i may copy the strategy of a selected counterpart j , which is chosen randomly among the N_i neighbors of i . The probability that i copies j 's strategy depends on the difference between the payoffs that they obtained in the previous

round of the game:

$$\mathcal{P}\{x_j^{(t)} \rightarrow x_i^{(t+1)}\} = \begin{cases} (\pi_j^{(t)} - \pi_i^{(t)})/\Phi & \text{if } \pi_j^{(t)} > \pi_i^{(t)}, \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

where Φ is a normalization constant that ensures $\mathcal{P}\{\cdot\} \in [0, 1]$. Note that because of the imitation mechanism of PI, the configurations $x_i = 1 \forall i$ and $x_i = 0 \forall i$ are absorbing states—the system cannot escape from them.

Best Response (BR) [25, 26] — Here players are fully rational and choose their strategy in order to maximize their payoff, given what their neighbors did in the last round. This means that each player i , given $x_{N_i}^{(t)}$, computes the payoffs that she would obtain by choosing action 1 (cooperating) or 0 (defecting) at time t , respectively $\tilde{\pi}_C^{(t)}$ and $\tilde{\pi}_D^{(t)}$. Then

$$x_i^{(t+1)} = \begin{cases} 1 & \text{if } \tilde{\pi}_C^{(t)} > \tilde{\pi}_D^{(t)}, \\ 0 & \text{if } \tilde{\pi}_C^{(t)} < \tilde{\pi}_D^{(t)}, \\ x_i^{(t)} & \text{if } \tilde{\pi}_C^{(t)} = \tilde{\pi}_D^{(t)}, \end{cases} \quad (7)$$

We use PI because it is equivalent, for a well-mixed population, to the well-known replicator dynamics [17], and BR because it is widely used in the economic literature. We then study how the system evolves starting from the initial random distribution of strategies. In particular, we want to check whether the system can reach a Nash equilibrium (a state where no player can increase her payoff by unilaterally changing her strategy). Note, however, that whereas a Nash equilibrium is stable by definition under BR dynamics, with PI this is not necessarily true: players can change action by copying better-performing neighbors, even if such change deteriorates their payoffs. Hence in the latter case a potential stationary state of the system does not necessarily correspond to a Nash equilibrium (unless the network is a complete graph [17]).

In what follows we present analytical and numerical results for the system described above. For simulations, without loss of generality we set $\rho_0 = 1/2$, $c = 1/2$ and $q = 1/10$ —but our results are valid for all values of these parameters in $(0, 1)$.¹ We will show averages over $\mathcal{N} = 20$ independent realizations of the system, an amount that we found enough to tame single-realization fluctuations (that are however shown with error bars, when visible). Note that in numerical simulations, when the system arrives to a Nash equilibrium, we change by hand the strategy of a random player in order to continue exploring other strategy configurations.

As stated in the introduction, we consider two representative kinds of population structure. Erdős-Rényi (ER) random graphs [19] are built by considering a collection of n nodes and adding a link between each pair of nodes with probability p . The resulting networks are homogeneous, with degree distribution decaying exponentially for large degree k and the average degree is $\bar{k} = (n - 1)p = m$. Scale-free (SF) random networks are generated using a configuration model [27] with a constraint $k_{max} < \sqrt{n}$, which gives rise to uncorrelated random networks [29] with degree distribution $P(k) \propto k^{-\gamma}$. The average degree depends on the network size n and converges to a finite value as n diverges²

$$\bar{k} = \frac{\gamma - 1}{\gamma - 2} \left[\frac{\sqrt{n}^{2-\gamma} - k_{min}^{2-\gamma}}{\sqrt{n}^{1-\gamma} - k_{min}^{1-\gamma}} \right] \xrightarrow{n \rightarrow \infty} \frac{\gamma - 1}{\gamma - 2} k_{min} =: m.$$

¹Note that we have to impose the condition $q < 1$ to avoid possible trapping into period-2 loops, where all players change action simultaneously at each round of the game, that can arise in anti-coordination under BR.

²In what follows we set $\gamma = 2.5$ and $k_{min} = 3$, and consider various sizes: $n = 1\,000$ ($\bar{k} = 6.415$), $n = 10\,000$ ($\bar{k} = 7.48$), $n = 100\,000$ ($\bar{k} = 8.13$), $n = 1\,000\,000$ ($\bar{k} = 8.51$). Note that for $n \rightarrow \infty$ it is $m := \bar{k}_\infty = 9$.

4 Results

We now present the results of the numerical simulations. We recall that in Ref. [22] we analyzed the evolutionary dynamics for the two games, under the same two evolutionary rules, by resorting to two analytical approaches, the homogeneous mean-field (MF) and the heterogeneous mean-field (HMF). While we refer to Ref. [22] for the complete analysis and detailed discussion, we include brief summaries of the analytical predictions in the following, as tools to understand the simulation results. Note that the MF approach, which should work best for homogeneous random graphs, deals with the aggregate density of cooperators $\rho(t)$, whereas, the HMF, which is more appropriate for heterogeneous networks, involves the computation of the quantity $\Theta(t) = \sum_k k P(k) \rho_k(t) / \bar{k}$ (*i.e.*, the average of the probability ρ_k that a node of degree k cooperates, weighted by the relative degree).

4.1 Best-shot game

Proportional Imitation

In this case, the MF prediction is that

$$\rho(t) = [1 + (\rho_0^{-1} - 1) \exp(cqt)]^{-1}, \quad 0 < \rho_0 < 1. \quad (8)$$

Hence the population converges to the state with no cooperators $\rho = 0$ (full defection), unless the initial state is $\rho_0 = 1$ (full cooperation). We see from Figure 1 that, also in simulations, the system always goes towards the absorbing state $x_i = 0 \ \forall i \Rightarrow \rho \equiv 0$ and the full dynamical evolution is in excellent agreement with the MF theory. The convergence toward full defection occurs because a defector cannot copy a neighboring cooperator (who has lower payoff by construction), whereas a cooperator will eventually copy one of her neighboring defectors (who has higher payoff). However, full defection is not a Nash equilibrium as any player surrounded by defectors would do better by cooperating.

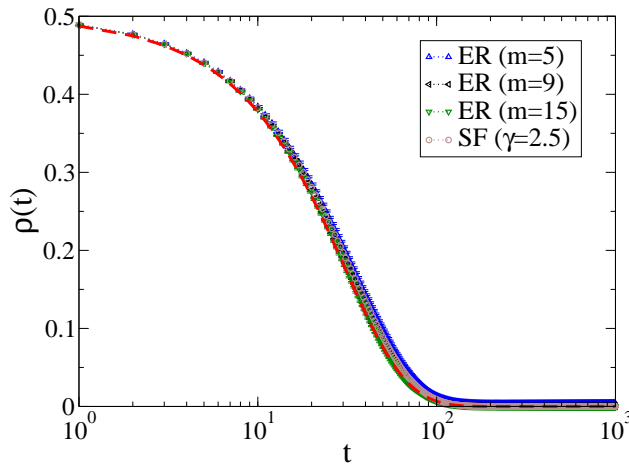


Figure 1: Best-shot games with Proportional Imitation: average $\rho(t)$ for Erdős-Rényi and scale-free graphs, with $n = 10^4$ (but results are independent on the specific value of n). The red dashed curve is the MF result—Eq.(8).

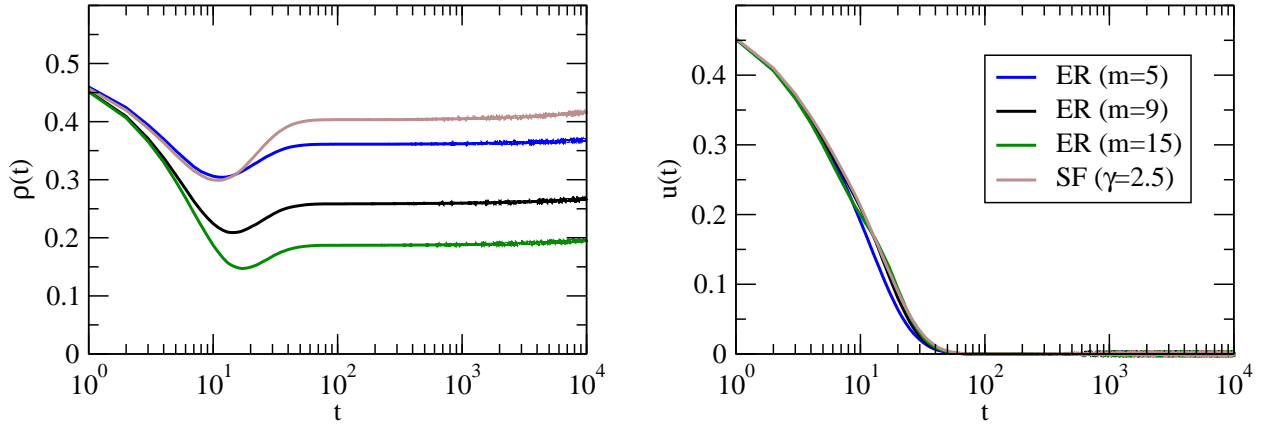


Figure 2: Best-shot games with Best Response: averages of $\rho(t)$ and fraction $u(t)$ of unsatisfied players (*i.e.*, who would obtain higher payoff by unilaterally switching action, and with $u = 0$ indicating the falling of the system into a Nash equilibrium) for Erdős-Rényi and scale-free graphs, with $n = 10^4$ (again, results are independent on the specific value of n).

In the case of scale-free networks, the behavior is again remarkably well described by the simple MF approach, which coincides with the HMF as $\rho_k(t = 0)$ does not depend on k .

Best Response

For the best-shot game with BR dynamics, the MF predicts that the final state is, for any initial condition, a mixed state $\rho = \rho_s$, where $\rho_s \in (0, 1)$ is the solution of the equation:

$$\rho_s = e^{-m\rho_s}. \quad (9)$$

The HMF approach leads to an analogous conclusion: the final state is Θ_s , which is the solution of the equation

$$\Theta_s = \sum_k (1 - \Theta_s)^k k P(k) / \bar{k}. \quad (10)$$

Since BR dynamics is guaranteed to lead to Nash equilibria, ρ_s and Θ_s are the attractors of the dynamics, their values depending only on the average degree of the network (in particular, they decrease for increasing network connectivity) but not on ρ_0 , c or q .

Numerical simulations indeed confirm such a picture: the dynamics finds a rich variety of Nash equilibria with intermediate cooperation level ρ^* (Figure 2); moreover, ρ^* decreases with increasing network connectivity (Figure 3)—in agreement with both Eq. (3) and Eq. (9). The key observation to understand the features of such equilibria is that, in best-shot games under BR dynamics, a player switches to defection as soon as one of her neighbors is a cooperator, and this happens with higher probability when the player has many social ties (see also Figure 4). The higher values of ρ^* in scale-free networks than in Erdős-Rényi graphs (of the same link density) can be explained by the presence, in the first case, of more low-degree nodes—who preferentially cooperate as they have few neighbors to exploit. However, low-degree nodes weight less in Θ ,

so that we find a Θ^* for random scale-free networks similar to the ρ^* of Erdős-Rényi random graphs (in agreement with the mean field predictions).³

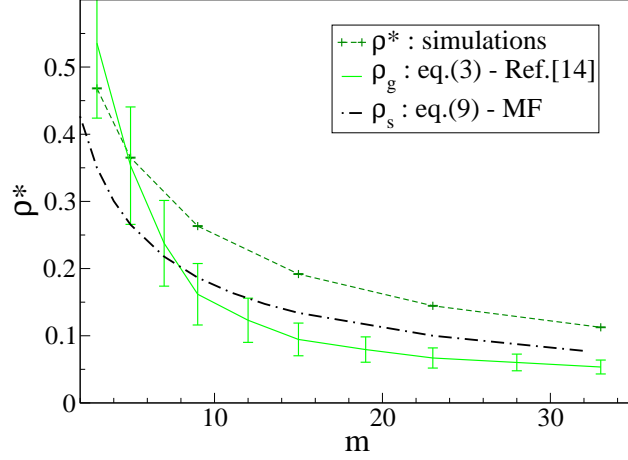


Figure 3: Best-shot games with Best Response on Erdős-Rényi random graphs: average ρ^* at Nash equilibria vs m from simulations ($n = 10^4$), theoretical prediction from Eq.(3), and MF Eq.(9).

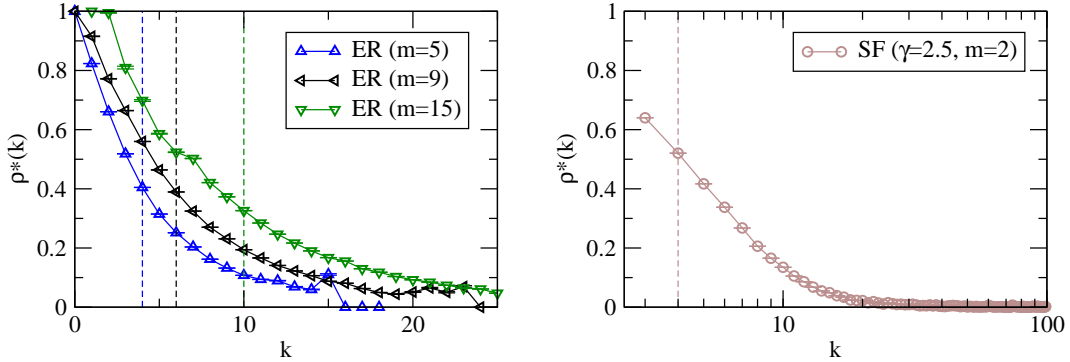


Figure 4: Best-shot games with Best Response: average $\rho^*(k)$ at Nash equilibria for nodes with different degrees k . The vertical dashed lines identify the thresholds τ from Eq.(3)—which basically depend on m .

The Nash equilibria found dynamically can be compared with those derived under the assumption of incomplete information by [14]—Eq.(3). We first inspect if the cooperation level of our equilibria lies in the range $\rho_g \in [\rho_{k<\tau}, \rho_{k\leq\tau}]$, where $\rho_{k<\tau}$ and $\rho_{k\leq\tau}$ are the densities of players with $k < \tau$ and $k \leq \tau$, respectively, and τ is given by Eq.(3). Figure 3 shows the average ρ^* at Nash equilibria for Erdős-Rényi random graphs as a function of m , obtained by numerical simulations, by the theoretical framework of [14], and by the mean field approach. Clearly, whereas the trends are similar, we observe that the cooperation levels of the static

³This is due to nodes with the highest degrees—that make the difference in $P(k)$ —not cooperating in the best-shot game, so that their effects on the system is negligible and $\rho_s \simeq \Theta_s$.

equilibria are compatible with simulations only for small values of m , while the mean field approximation works better for high average degrees. We then check whether $\rho^*(k)$, *i.e.*, the strategy profile as a function of the degree k of dynamically-found Nash equilibria, is compatible with $\sigma(k)$ from Eq. (3). Figure 4 shows that, as predicted in [14], $\rho^*(k)$ is generally non-increasing; however, the behavior is not step-like. Summing up, the dynamics leads to Nash equilibria which share some qualitative features but do not coincide with those identified in [14] under the assumption of incomplete information. Note that this does not occur because the configuration space is not sufficiently explored⁴, but rather because the two approaches are intrinsically different and do not have necessarily to give the same results. In fact, the equilibria that are evolutionary selected by our deterministic BR-based dynamics are indeed proper Nash, whereas, the framework of [14] is probabilistic and thus selects Bayes-Nash equilibria.

4.2 Coordination game

For coordination games the picture is richer, as the behavior of the system depends on the ratio α/c . Without loss of generality, we leave $c = 1/2$ fixed and vary the value of α in the range $(0, c)$.

Proportional imitation

The MF theory for the coordination game with PI predicts the existence of a critical value $\alpha_c = c/(m\rho_0)$, such that the final state of the dynamics is $\rho = 0$ (full defection) when $\alpha < \alpha_c$, and $\rho = 1$ (full cooperation) when $\alpha > \alpha_c$. Note, however, that while full defection is always a Nash equilibrium for the coordination game, full cooperation becomes a Nash equilibrium only when $\alpha > c/k_{min}$, where k_{min} is the smallest degree in the network—which means that only networks with $k_{min} > c/\alpha > 1$ may feature a fully cooperative Nash equilibrium. For Erdős-Rényi graphs, numerical simulations agree with this picture (Figure 5), as a discontinuous transition for $\alpha = \alpha_T$ is observed. The value α_T of the transition point found numerically is smaller than the MF prediction α_c ; however $\alpha_T \rightarrow \alpha_c$ as the average degree m grows. Note that apart from full defection, no other Nash equilibrium (with intermediate cooperation levels) is found.

The HMF theory for PI dynamics predicts a discontinuous transition between a stable fully defective equilibrium $\Theta = 0$ for $\alpha < \alpha_c$ and full cooperation $\Theta = 1$ for $\alpha > \alpha_c$ [22]. The critical value is now $\alpha_c = c/\Theta_2$, where

$$\Theta_2 := \sum_k k^2 P(k) \rho_k / \bar{k}. \quad (11)$$

The quantity Θ_2 is related to the second moment of the degree distribution $\langle k^2 \rangle$ and it can be shown to diverge for networks with $\gamma < 3$ as the system size n goes to infinity. Therefore, at odds with the case of Erdős-Rényi random graphs, for scale-free networks the threshold $\alpha_c \rightarrow 0$ as the system size n diverges. This vanishing of the transition point is analogous to what occurs for other processes on scale-free networks, such as percolation of epidemic spreading [30]. This phenomenon is due to the presence of hubs, *i.e.*, the nodes with very large degree (diverging with the network size n). For any value of $\alpha > 0$, the payoffs π_C of cooperating for the largest hubs will become positive for sufficiently large values of n . Those hubs then spread the cooperative strategy to the rest of the network.

⁴We have tried to manually drive the system into one of the equilibria derived by [14] with $\sigma(k) = 1$ for all $k < \tau$, $\sigma(k) = 0$ for all $k > \tau$, $\sigma(\tau) \in \{0, 1\}$. To do so, we set at the beginning of the evolution a configuration with the action of players with $k < \tau$ frozen to 1, the action of players with $k > \tau$ frozen to 0, and the action of players with $k = \tau$ free to evolve according to BR. Across many realizations, the evolution never converges to a Nash equilibrium—what happens instead is that all players with $k = \tau$ became satisfied (*i.e.*, they cannot increase their payoffs by unilaterally changing action), but making unsatisfied the others who cannot change action.

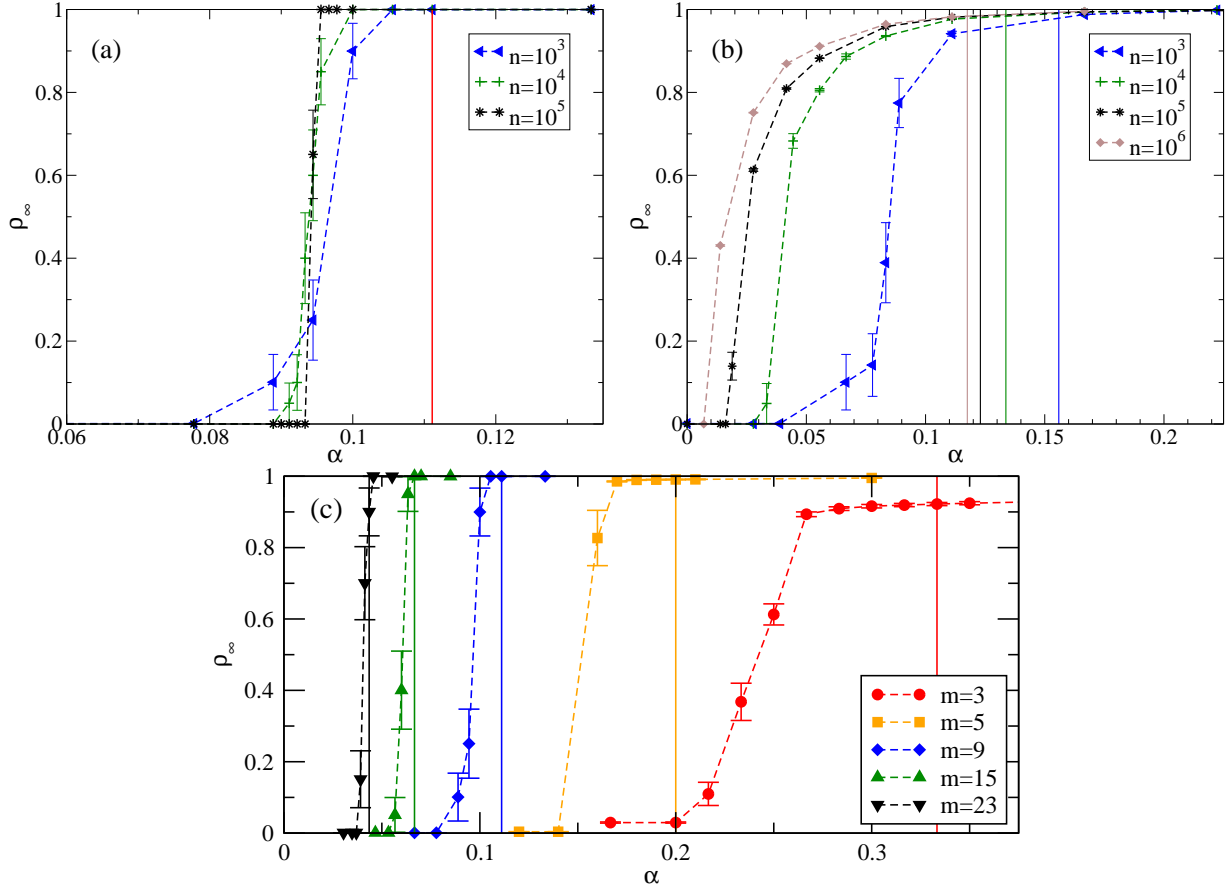


Figure 5: Coordination games with Proportional Imitation. The vertical solid lines identify the values of α_c . Erdős-Rényi random graphs: (a) stationary cooperation levels ρ_∞ vs α for $m = 9$ and various n ; (c) stationary cooperation levels ρ_∞ vs α for $n = 10^3$ and various m . Note that for small values of m such that $p = m/(n-1) < \ln(n)/n$ the graph is disconnected: isolated nodes are bound to their initial action, so that full defection and full cooperation are not accessible states; additionally, defective behavior spreads easily inside isolated components which are poorly connected (also for high values of α), so that ρ_∞ remains far from 1. Scale-free networks: (b) stationary cooperation levels ρ_∞ vs α for $m = 9$ and various n .

Numerical simulations of PI dynamics on scale-free networks confirm only in part the theoretical picture (Figure 5). We observe a continuous transition between a fully defective Nash equilibrium, for small values of α and final states with intermediate cooperation (which are not Nash equilibria) for $\alpha > \alpha_T$. As α keeps increasing, cooperation becomes the stable strategy for an increasing number of nodes, and the system heads smoothly towards full cooperation—which in this case becomes a stable Nash equilibrium starting from $\alpha = c/k_{min}$ (which is finite as $k_{min} > 0$ in the configuration model). The transition point α_T tends to zero as the system size grows. Hence HMF theory predicts correctly that the fully defective state disappears in the large size limit (a phenomenon not captured by MF or by the one-shot results in [14]), but it fails in

predicting that the transition is continuous.

Best Response

For coordination games under BR, the approximate MF calculations predict again the existence of a critical value $\alpha_c = c/(m\rho_0)$ such that for $\alpha \ll \alpha_c$ the only Nash equilibrium is full defection, whereas, for $\alpha \gg \alpha_c$ the final state exhibits a large level ρ^* of cooperation. Note that here any Nash equilibrium features players with $k < c/\alpha$ being defectors by construction; therefore, the fully cooperative Nash equilibrium is achieved for $\alpha \gg \alpha_c$ but only in networks with $k_{min} > 1$.

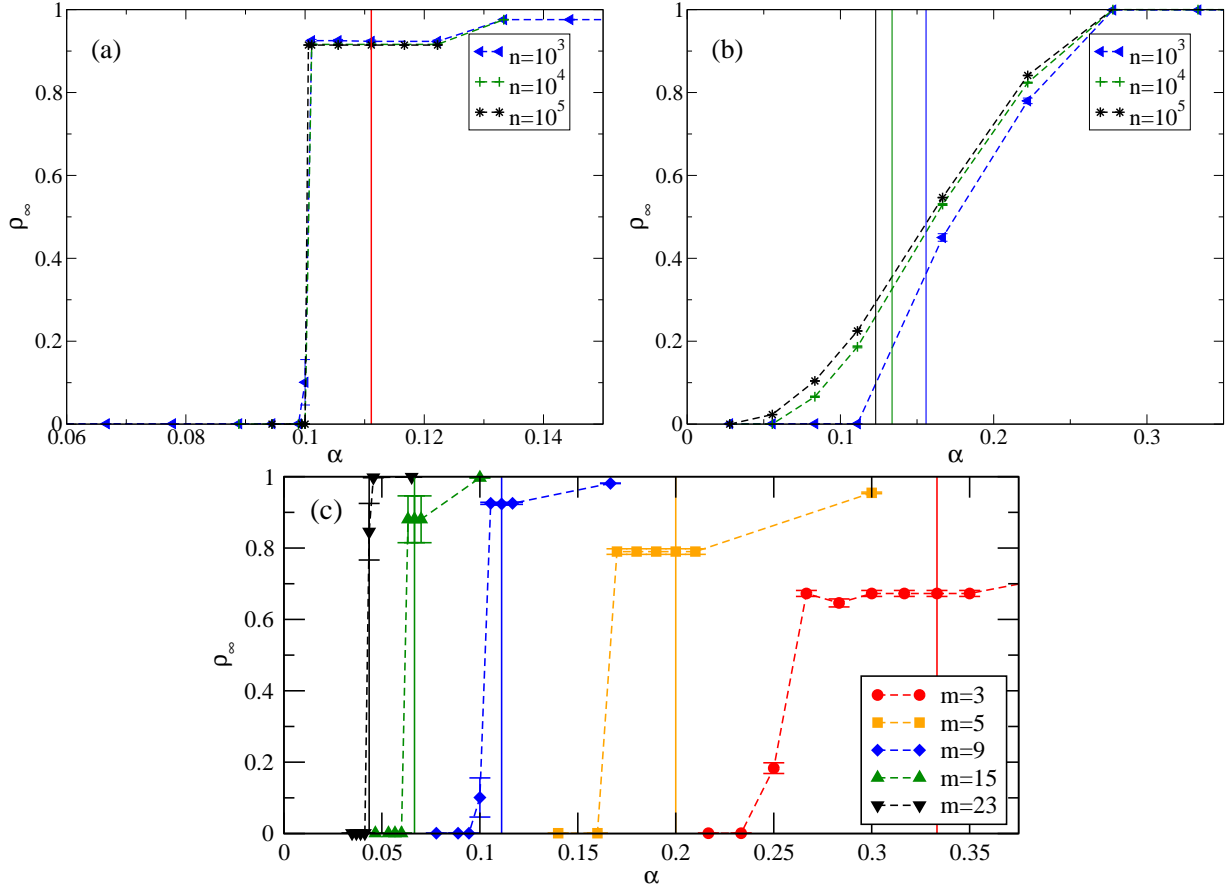


Figure 6: Coordination games with Best Response. The vertical solid lines identify the values of α_c . Erdős-Rényi random graphs: (a) stationary cooperation levels ρ_∞ vs α for $m = 9$ and various n ; (c) stationary cooperation levels ρ_∞ vs α for $n = 10^3$ and various m . Note that as m decreases, the amount of nodes with $k < c/\alpha$ increases; as these players are defectors by construction, $\rho_\infty \not\rightarrow 1$ even for $\alpha \rightarrow \infty$. Scale-free networks: (b) stationary cooperation levels ρ_∞ vs α for $m = 9$ and various n .

The behavior found in numerical simulations is similar to the case of PI (Figure 6). On Erdős-Rényi

random graphs a discontinuous transition is found (players have similar degrees, and thus, for a given ρ , they become cooperators for similar values of α). An important difference is that many non-trivial Nash equilibria (with intermediate cooperation levels ρ^*) are found just above the transition point. A continuous transition is found instead for scale-free networks. Here players have different degrees, and for a given ρ each degree class requires its own value of α to switch to cooperation.

The HMF approach yields a self-consistent equation for the equilibrium Θ_s :

$$\Theta_s = \sum_{k > c/(\alpha\Theta_s)} kP(k)/\bar{k} \quad (12)$$

If the network is scale-free with $2 < \gamma < 3$, Θ_s represents a stable equilibrium whose dependence on α is of the form $\Theta_s \sim \alpha^{(\gamma-2)/(3-\gamma)}$, *i.e.*, there exists a non-vanishing cooperation level Θ_s no matter how small the value of α . However, if the network is homogeneous (*e.g.*, $\gamma > 3$), Θ_s becomes unstable and for $\alpha \rightarrow 0$ the system always falls in the fully defective Nash equilibria.

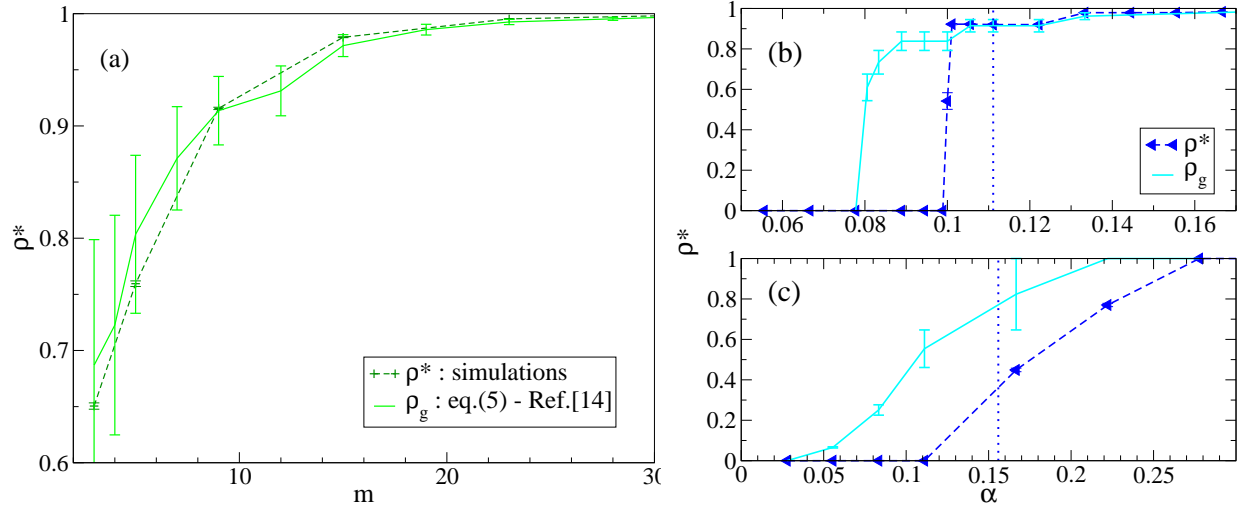


Figure 7: Coordination games with Best Response: average ρ^* at Nash equilibria from simulations and theoretical prediction from Eq.(5). (a) $\rho^*(m)$ for Erdős-Rényi random graphs with $n = 10^4$ at $\alpha = \alpha_c$; (b) $\rho^*(\alpha)$ for Erdős-Rényi random graphs with $n = 10^3$ and $m = 9$; (c) $\rho^*(\alpha)$ for scale-free networks with $n = 10^3$ and $m = 9$. The vertical dashed lines denote the critical value α_c .

The existence, in some intervals of α values, of Nash equilibria with intermediate levels of cooperation calls for the comparison between the simulation results and the predictions of Ref. [14]. Figure 7(a) shows that, for Erdős-Rényi random graphs at $\alpha = \alpha_c$, the cooperation level of equilibria found dynamically lies in the range $\rho_g \in [\rho_{k \geq \tau}, \rho_{k > \tau}]$, where τ is given by Eq.(5). However, we do not find such a good agreement for all values of α nor for scale-free networks (Figures 7b,c).

Another important prediction of the HMF approach is that $\rho_k \rightarrow 0$ when $k < c/(\alpha\Theta_s)$, and $\rho_k \rightarrow 1$ for $k > c/(\alpha\Theta_s)$. In this sense, the equilibria predicted by HMF agree qualitatively with those found in [14]: players' actions show a non-decreasing dependence on their degrees. Indeed, Figure 8 shows that the average cooperation level $\rho^*(k)$ of Nash equilibria found dynamically in simulations for $\alpha = \alpha_c$ is generally non-decreasing in k , and that a step-like behavior of $\sigma(k)$, as predicted in [14], becomes a good approximation

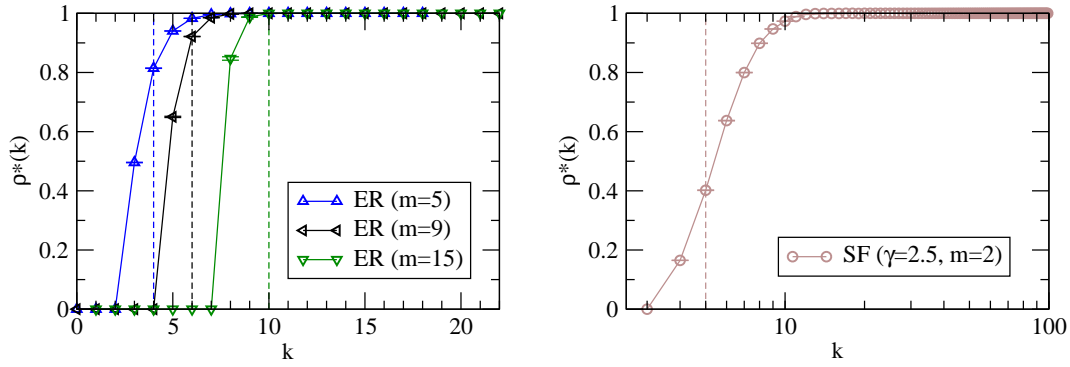


Figure 8: Coordination games with Best Response: average $\rho^*(k)$ at Nash equilibria at $\alpha = \alpha_c$ for nodes with different degrees k . The vertical dashed lines identify the thresholds τ from Eq.(5).

for large n . In conclusion, the equilibria predicted in [14], are a good approximation of the equilibria found dynamically only in the thermodynamic limit ($n \rightarrow \infty$) and, more importantly, only for a small subset of α values.

4.3 The effect of correlated networks

In order to study the effects of topologies with degree-degree correlations on the behavior of the games, we run simulations with interaction patterns given by scale-free networks with assortative and disassortative correlations. In particular, we consider networks generated using the prescriptions of the model by Weber and Porto [28], again with the constraint $k_{max} < \sqrt{n}$ on the largest degree. The correlation properties of the resulting topologies are dependent on the model parameter β . The average degree of the nearest-neighbors of a node of degree k , $k_{nn}(k) := \sum_j jP(j|k)$, is proportional to k^β , so that for $\beta > 0$ neighbors of nodes with large k have large degree (assortative networks), while for $\beta < 0$ nodes with large k have neighbors with small degree and vice versa (disassortative networks). The uncorrelated case is recovered for $\beta = 0$.

For the best-shot game (Fig. 9), the simple case of PI dynamics is totally unaffected by the presence of correlations: the temporal evolution and the asymptotic state do not depend at all on β . Instead, when strategies evolve according to BR, the dynamics leads to Nash equilibria with levels ρ^* of cooperation (dominated by the behavior of low-degree nodes) that decrease with β . The reason is that for $\beta < 0$ low-degree nodes are connected to hubs, which are likely to be defectors; as a consequence they tend to cooperate and ρ^* is higher. The opposite occurs in assortative networks ($\beta > 0$): low-degree nodes are connected to each other and less cooperation is needed.

For coordination games with PI dynamics, the effect of assortative (disassortative) correlations is to make broader (sharper) the transition observed as a function of α for finite network size n (Fig. 10). This can be understood by considering that, in the assortative case, hubs tend to be connected with each other, so that cooperation can spread more easily among them and be sustained by mutual connections: the critical density of stable cooperators decreases. At the same time, low-degree nodes are connected only with each other, so that they require larger values of α to sustain cooperation within themselves (as compared to the uncorrelated case). As a result, the transition becomes broader. An analogous argument explains the sharper transition for disassortative networks. In the case of BR dynamics, the same picture applies.

In conclusion, degree-degree correlations of the interaction pattern do not change the qualitative features

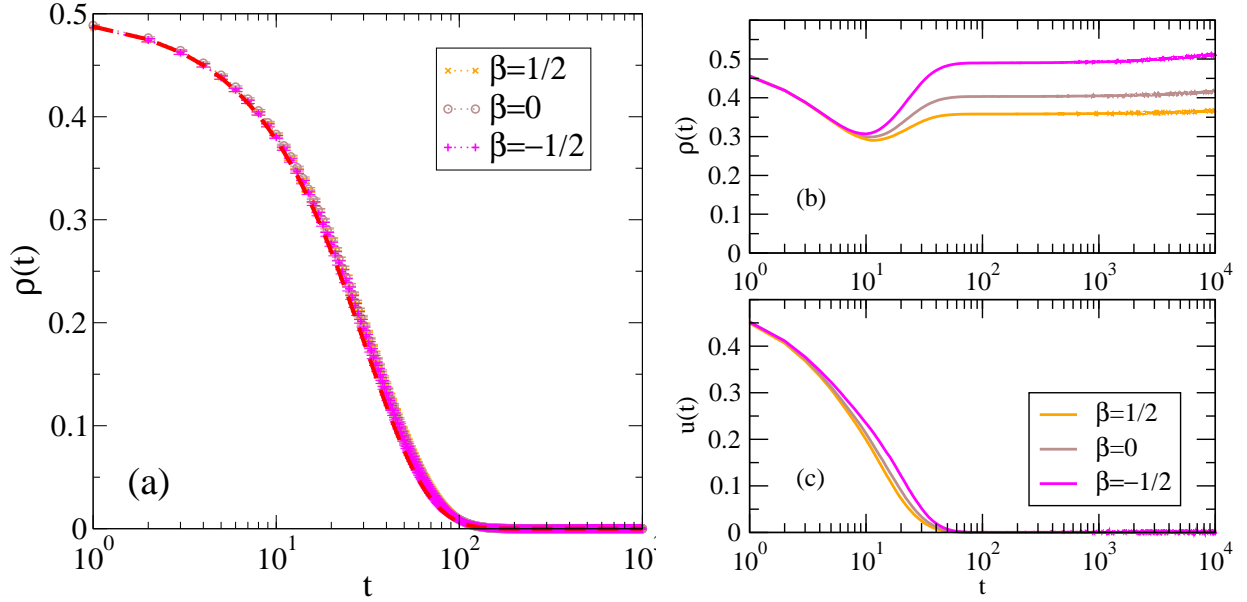


Figure 9: Best-shot games for correlated networks with $m = 9$, $n = 10^4$ and various β . Again, these results are independent on the specific value of n . Proportional Imitation: (a) $\rho(t)$ —the red dashed curve being the MF Eq.(8). Best Response: (b) $\rho(t)$ and (c) fraction $u(t)$ of unsatisfied players (with $u = 0$ indicating the falling of the system into a Nash equilibrium).

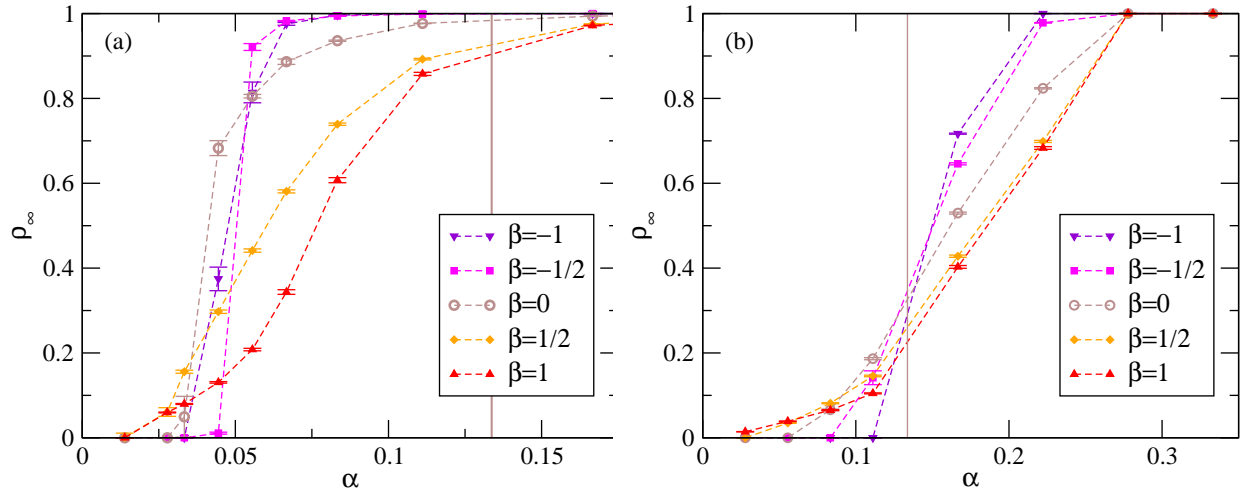


Figure 10: Coordination games with PI (a) and BR (b) for correlated networks with $m = 9$, $n = 10^4$ and various β . Again, these results are independent on the specific value of n . The vertical lines identify the values of α_c .

of the behavior of games on scale-free networks.

5 Summary and Conclusion

In this work we have studied by numerical simulations two kinds of games, namely the best-shot and the coordination games, as representatives of two wide classes of social interactions (strategic substitutes and strategic complements, respectively). In these games, the welfare of a player depends both on her own actions and on the actions taken by her partners; thus, we have considered different topological structures for the pattern of interactions. We have embedded these games into an evolutionary framework, described by two types of dynamics, widely employed to model evolving populations: Proportional Imitation, in which players imitate more successful neighbors, and Best Response, in which players are rational and make optimal choices. We have performed numerical simulations, determining the attractors of the evolution, and characterizing them in term of Nash equilibria. By comparison with the results of numerical simulations we have assessed the validity of the mean field approaches in describing such systems, and we have also compared our findings with the theoretical predictions of [14] about the features of Nash equilibria in one-shot games under incomplete information.

Generally, we observe that the behavior of the system is highly influenced by the dynamics employed and by the population structure. Strategic substitutes under PI dynamics represents the simplest case, in which full defection is the only accessible (but non-Nash) equilibrium, whatever the underlying topology, in complete agreement with the mean field predictions. This suggests that the failure to find a Nash equilibrium arises from the (bounded rational) dynamics and, with the benefit of hindsight, it is clear that imitation is not a good procedure for players to decide in anti-coordination games. Such a conclusion is supported by the behavior observed for strategic substitutes under BR dynamics. Indeed, here we observe many stable Nash equilibria, with cooperation level ρ^* slightly smaller than (but close to) the mean field prediction ρ_c . Moreover, in simulations we observe precisely what is predicted by the MF theory, namely that ρ^* decreases with increasing network connectivity and does not depend on the initial conditions, game and simulation parameters, and system size (which was taken as infinite in the analytical calculations). Concerning the topology, ρ^* is enhanced in heterogeneous networks because of more low-degree nodes who are typically cooperators. Additionally, disassortativity has a positive effect on the cooperation levels, as low-degree nodes are connected to high-degree nodes who are likely to be defectors, and thus tend to cooperate even more. Vice versa, assortativity allows more low-degree nodes to defect.

The picture is far more rich and interesting for strategic complements, that feature an additional parameter α playing a key role in determining which equilibria are dynamically accessible. Indeed, in the case of PI for Erdős-Rényi random graphs we observe two kinds of stationary states: fully defective Nash equilibria for $\alpha < \alpha_T$, and full cooperation for $\alpha > \alpha_T$ (that becomes Nash equilibrium only when $\alpha > c/k_{min}$). Remarkably, $\alpha_T \rightarrow \alpha_c$ for $n, m \rightarrow \infty$, where $\alpha_c = c/(m\rho_0)$ is the value predicted by the MF theory. We thus see that imitation is indeed a good procedure to choose actions in a coordination setup: PI does lead to Nash equilibria, and indeed it makes a very precise prediction: a unique equilibrium that depends on the initial density. If the topology is scale-free, HMF theory and simulations agree on α_T and α_c going to zero for $n \rightarrow \infty$, indicating that in these cases cooperation emerges also when the incentive to cooperate vanishes. The situation with BR dynamics is rather similar to the case of PI, with the single difference that for Erdős-Rényi random graphs and $\alpha > \alpha_c$ the stationary state is now a Nash equilibrium, and again full cooperation is achieved for $\alpha > c/k_{min}$. Thus we see that, with BR, equilibria with intermediate values of the density of cooperators are obtained in a range of initial densities. Compared to the situation with PI, in which we only find the absorbing states as equilibria, this points to the fact that more rational players can

eventually converge to equilibria with higher payoffs. Finally, we have found that, whatever the dynamics, the presence of topological correlations in the network does not change the global qualitative picture.

Besides the general features of the dynamics, we have also been able to study the degree-dependent features of Nash equilibria (when present) and compare our findings with the theoretical predictions of [14], in the which authors consider games played once and for all but using only partial information on the underlying topology. Our conclusion is that, while the Nash equilibria predicted in [14] cannot (always) be reached, their theory still provides good guidance on the shape and features of the Nash equilibria which are evolutionarily accessible.

Finally, it is interesting to note that, as in [31, 32], we find that the outcome of the evolution depends on the dynamics and the network properties. Remarkably, here we go beyond those works as our study of individual behaviors allows us to make a connection with the results obtained in a more traditional economic framework.

References

- [1] Katz E and Lazarsfeld PF, *Personal Influence: The Part Played by People in the Flow of Mass Communication* (Glencoe: Free Press, 1955).
- [2] Coleman J, *Medical Innovation: A Diffusion Study* (New York: Bobbs-Merrill, 1966).
- [3] Granovetter M, *Getting a Job: A Study of Contacts and Careers* (Evanston: Northwestern University Press, 1994).
- [4] Foster AD and Rosenzweig MR, *Journal of Political Economy* 103 (6), 1176-1209 (1995).
- [5] Glaeser E, Sacerdote B and Scheinkman J, *Quarterly Journal of Economics* 111, 507-548 (1996).
- [6] Topa G, *Review of Economic Studies* 68 (2), 261-295 (2001).
- [7] Conley T and Udry C, *American Economic Review* 100(1), 35-69 (2010).
- [8] Vega-Redondo F, *Complex Social Networks*, in *Econometric Society Monograph Series* (Cambridge: Cambridge University Press, 2007).
- [9] Goyal S, *Connections: An Introduction to the Economics of Networks* (Princeton: Princeton University Press, 2007).
- [10] Jackson MO, *Social and Economic Networks* (Princeton: Princeton University Press, 2008).
- [11] Carlsson H and van Damme E, *Econometrica* 61 (5), 989-1018 (1993).
- [12] Morris S and Shin H, *Global Games: Theory and Applications*, in *Advances in Economics and Econometrics: Proceedings of the Eighth World Congress of the Econometric Society* 56-114 (Cambridge: Cambridge University Press, 2003).
- [13] Bramoullé Y and Kranton R, *Journal of Economic Theory* 135 (1), 478-494 (2007).
- [14] Galeotti A, Goyal S, Jackson MO, Vega-Redondo F and Yariv L, *Review of Economic Studies* (2010) 77, 218-244 (2010).
- [15] Maynard Smith J and Price GR. *Nature* 246, 15-18 (1973).
- [16] Hofbauer J and Sigmund K, *Evolutionary Games and Population Dynamics* (Cambridge UK: Cambridge University Press, 1998).
- [17] Gintis H, *Game Theory Evolving* (Princeton, NJ: Princeton University Press, 2009).
- [18] Boncinelli, L and Pin P. *Games and Economic Behavior* 75 (2), 538-554 (2012).
- [19] Erdős SP and Rényi A, *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* 5, 17-61 (1960).
- [20] Barabási AL and Albert R, *Science* 286, 509-512 (1999).
- [21] Bulow J, Geanakoplos J and Klemperer P, *Journal of Political Economy* 93, 488-511 (1985).
- [22] Cimini G, Castellano C and Sánchez A, arXiv preprint (2014).

- [23] Pastor-Satorras R and Vespignani A, *Physical Review Letters* 86, 3200-3203 (2001).
- [24] Helbing D, *Physica A* 181, 29-52 (1992).
- [25] Matsui A, *Journal of Economic Theory* 57, 343-362 (1992).
- [26] Blume LE, *Games and Economic Behavior* 5, 387-424 (1993).
- [27] Newman MEJ, *SIAM Review* 45, 167-256 (2003).
- [28] Weber S and Porto M, *Physical Review E* 76, 046111 (2007).
- [29] Catanzaro M, Boguñá M and Pastor-Satorras R, *Physical Review E* 71, 027103 (2005).
- [30] Dorogovtsev S N, Goltsev A V and Mendes J F F, *Reviews of Modern Physics* 80, 1275 (2008).
- [31] Roca CP, Cuesta J and Sánchez A, *Physical Review E* 80, 046106 (2009).
- [32] Roca CP, Cuesta J and Sánchez A, *Physics of Life Reviews* 6, 208-249 (2009).